Observation on the Non-Homogeneous Binary Quadratic Diophantine Equation $5x^2 - 2y^2 = 27$

A. Kavitha
Assistant Professor, Department Mathematics, Shrimati Indra Gandhi College, Trichy-620002, India

B. Kiruhtika
M.phil., Scholar, Department Mathematics, Shrimati Indra Gandhi College, Trichy-620002, India

Abstract – The quadratic Diophantine equation given by $5x^2 - 2y^2 = 27$ is studied for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

Index Terms – Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.

1. INTRODUCTION

The hyperbola represented by the Diophantine equations of the form $ax^2 + by^2 = N, (a, b \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. For an extensive review, one may refer [1-16].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation $5x^2 - 2y^2 = 27$ given by representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

2. METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$5x^2 - 2y^2 = 27 \quad (1)$$

It is to be noted that (1) represent a hyperbola

Taking $x = X + 2T$, $y = X + 5T$ (2)

in (1), it reduces to the equation,

$$X^2 = 10T^2 + 9 \quad (3)$$

The smallest positive integer solution (T_0, X_0) of (3) is

$$T_0 = 2, \; X_0 = 7$$

To obtain, the other solutions of (3), consider the Pellian equation

$$X^2 = 10T^2 + 1 \quad (4)$$

whose smallest positive integer solution is

$\tilde{T}_o = 6$, $\tilde{X}_0 = 19$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (4) is given by,

$$\tilde{X}_n + \sqrt{10}\tilde{T}_n = \left(19 + 6\sqrt{12}\right)^{n+1}, \text{ where } n = 0,1,2,\ldots \quad (5)$$

Since, irrational roots occur in pairs, we have,

$$\tilde{X}_n - \sqrt{10}\tilde{T}_n = \left(19 - 6\sqrt{10}\right)^{n+1}, \text{ where } n = 0,1,2,\ldots \quad (6)$$

From (5) and (6) solving for $(\tilde{T}_n, \tilde{X}_n)$ we have,

$$\tilde{X}_n = \frac{1}{2}\left[\left(19 + 6\sqrt{10}\right)^{n+1} + \left(19 - 6\sqrt{10}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{10}}\left[\left(19 + 6\sqrt{10}\right)^{n+1} - \left(19 - 6\sqrt{10}\right)^{n+1}\right] = \frac{1}{2\sqrt{12}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution (T_{n+1}, X_{n+1}) of (3) is found to be

$$T_{n+1} = X_0\tilde{T}_n + T_0\tilde{X}_n$$

$$T_{n+1} = \frac{7}{2\sqrt{10}} g_n + \frac{2}{2} f_n \quad (7)$$

$$X_{n+1} = X_0\tilde{X}_n + 10T_0\tilde{T}_n \quad (8)$$

Using (7) and (8) in (2), we have

$$x_{n+1} = X_{n+1} + 2T_{n+1} = 11 f_n + \frac{34}{2\sqrt{10}} g_n \qquad (9)$$

$$y_{n+1} = X_{n+1} + 5T_{n+1} = \frac{17}{2} f_n + \frac{55}{2\sqrt{10}} g_n \qquad (10)$$

Thus, (9) and (10) represent the integer solutions of the hyperbola (1)

A few numerical examples are given in the following Table 1

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	11	17
0	413	653
1	15683	24797
2	595541	941633
3	22614875	3575725

Recurrence relations for x and y are:

$$x_{n+3} - 38x_{n+2} + x_{n+1} = 0, n = -1,0,1,\ldots$$

$$y_{n+3} - 38y_{n+2} + y_{n+1} = 0, n = -1,0,1,\ldots.$$

2.1. A few interesting relations among the solutions are given below

- $12y_{n+1} + 19x_{n+1} + x_{n+2} = 0$
- $12y_{n+2} + x_{n+1} - 19x_{n+2} = 0$
- $12y_{n+3} + 19x_{n+1} + 721x_{n+2} = 0$
- $456y_{n+3} + 721x_{n+1} - x_{n+3} = 0$
- $24y_{n+2} + x_{n+1} - x_{n+3} = 0$
- $456y_{n+3} + 721x_{n+2} - x_{n+3} = 0$
- $y_{n+2} - 30x_{n+1} - 19y_{n+1} = 0$
- $y_{n+3} - 1140y_{n+1} - 721y_{n+1} = 0$
- $19y_{n+3} - 30x_{n+1} - 12y_{n+3} = 0$
- $12y_{n+1} + 721x_{n+2} - 19x_{n+3} = 0$
- $12y_{n+2} + 19x_{n+2} - x_{n+3} = 0$
- $12y_{n+3} + x_{n+2} - 19x_{n+3} = 0$
- $y_{n+3} - 30x_{n+2} - 19y_{n+2} = 0$
- $19y_{n+1} + 1140x_{n+2} - 19y_{n+3} = 0$
- $721y_{n+3} - 1140x_{n+2} - y_{n+1} = 0$
- $19y_{n+3} - 30x_{n+3} - y_{n+1} = 0$
- $y_{n+3} - 721y_{n+2} + 12y_{n+1} = 0$
- $30x_{n+3} - 721y_{n+2} + 12y_{n+1} = 0$

2.2. Each of the following expression represents a cubic integer

- $\frac{1}{324}\left[2612x_{3n+3} - 68x_{3n+4} + 7836x_{n+1} - 204x_{n+2}\right]$
- $\frac{1}{12312}\left[99188x_{3n+3} - 68x_{3n+5} + 297564x_{n+1} - 204x_{n+3}\right]$
- $\frac{1}{27}\left[110x_{3n+3} - 68y_{3n+3} + 330x_{n+1} - 204y_{n+1}\right]$
- $\frac{1}{513}\left[4130x_{3n+3} - 68y_{3n+4} + 12390x_{n+1} - 204y_{n+2}\right]$
- $\frac{1}{19467}\left[156830x_{3n+3} - 68y_{3n+5} + 470490x_{n+1} - 204y_{n+3}\right]$
- $\frac{1}{324}\left[99188x_{3n+4} - 2612x_{3n+5} + 297564x_{n+2} - 7836x_{n+3}\right]$
- $\frac{1}{513}\left[110x_{3n+4} - 2612y_{3n+3} + 330x_{n+2} - 7836y_{n+1}\right]$
- $\frac{1}{27}\left[4130x_{3n+4} - 2612y_{3n+3} + 12390x_{n+2} - 7836y_{n+2}\right]$
- $\frac{1}{513}\left[156830x_{3n+4} - 2612y_{3n+5} + 470490x_{n+2} - 7836y_{n+3}\right]$
- $\frac{1}{19467}\left[110x_{3n+5} - 99188y_{3n+3} + 330x_{n+3} - 297564y_{n+1}\right]$
- $\frac{1}{513}\left[4130x_{3n+5} - 99188y_{3n+4} + 12390x_{3n+3} - 297564y_{n+2}\right]$
- $\frac{1}{27}\left[156830x_{3n+5} - 99188y_{3n+3} + 470490x_{n+3} - 297564y_{n+3}\right]$
- $\frac{1}{810}\left[110y_{3n+4} - 4130y_{3n+3} + 330y_{n+2} - 12390y_{n+1}\right]$

- $\frac{1}{30780}\left[110y_{3n+5}-156830y_{3n+3}+330y_{n+3}-470490y_{n+1}\right]$
- $\frac{1}{810}\left[4130y_{3n+5}-156830y_{3n+4}+12390y_{n+3}-156830y_{n+2}\right]$

2.3. Each of the following expression represents a bi-quadratic

1. $\frac{1}{(324)^2}\left[\begin{array}{l}274197312x_{4n+4}-7138368x_{4n+5}\\+4(2612x_{n+1}-68x_{n+2})^2-209952\end{array}\right]$
2. $\frac{1}{(12312)^2}\left[\begin{array}{l}1221202656x_{4n+4}-837216x_{4n+6}\\+4(99188x_{n+1}-68x_{n+3})^2\\-303170688\end{array}\right]$
3. $\frac{1}{(27)^2}\left[\begin{array}{l}2970x_{4n+4}-1836y_{4n+4}\\+4(110x_{n+1}-68y_{n+1})^2-1458\end{array}\right]$
4. $\frac{1}{(513)^2}\left[\begin{array}{l}2118690x_{4n+4}-34884y_{4n+5}\\+4(4130x_{n+1}-68y_{n+2})^2-526338\end{array}\right]$
5. $\frac{1}{(19467)^2}\left[\begin{array}{l}3044249460x_{4n+4}-1323756y_{4n+6}\\+4(156830x_{n+1}-68x_{n+3})^2\\-757928178\end{array}\right]$
6. $\frac{1}{(324)^2}\left[\begin{array}{l}32136912x_{4n+5}-846288x_{4n+6}\\+4(99188x_{n+2}-2612x_{n+3})^2-209952\end{array}\right]$
7. $\frac{1}{(513)^2}\left[\begin{array}{l}56430x_{4n+5}-1339956y_{4n+4}\\+4(110x_{n+2}-2612y_{n+1})^2-526338\end{array}\right]$
8. $\frac{1}{(27)^2}\left[\begin{array}{l}111510x_{4n+5}-70524y_{4n+5}\\+4(4130x_{n+2}-2612y_{n+2})^2-1458\end{array}\right]$
9. $\frac{1}{(513)^2}\left[\begin{array}{l}80453790x_{4n+5}-1339956y_{4n+6}\\+4(15830x_{n+2}-2612y_{n+3})^2-526338\end{array}\right]$
10. $\frac{1}{(19467)^2}\left[\begin{array}{l}2141370x_{4n+6}-1930892796y_{4n+4}\\+4(110x_{n+3}-99188y_{n+1})^2-757928178\end{array}\right]$
11. $\frac{1}{(513)^2}\left[\begin{array}{l}2118690x_{4n+6}-50883444y_{4n+5}\\+4(4130x_{n+3}-99188y_{n+2})^2-526338\end{array}\right]$
12. $\frac{1}{(27)^2}\left[\begin{array}{l}4234410x_{4n+6}-2678076y_{4n+6}\\+4(156830x_{n+3}-99188y_{n+3})^2-1458\end{array}\right]$
13. $\frac{1}{(810)^2}\left[\begin{array}{l}89100y_{4n+5}-3345300y_{4n+4}\\+4(-4130y_{n+1}+110y_{n+2})^2-1312200\end{array}\right]$
14. $\frac{1}{(30780)^2}\left[\begin{array}{l}385800y_{4n+6}-4827227400y_{4n+4}\\+4(-156830y_{n+1}+110y_{n+3})^2\\-1894816800\end{array}\right]$
15. $\frac{1}{(810)^2}\left[\begin{array}{l}3345300y_{4n+6}-127032300y_{4n+5}\\+4(-156830y_{n+2}+4130y_{n+3})^2-1312200\end{array}\right]$

2.4. Each of the following expression represents a quintic integer

- $\frac{1}{(324)^3}\left[\begin{array}{l}274197312x_{5n+5}-7138368x_{5n+6}\\+5(2612x_{n+1}-68x_{n+2})^3-4231440x_{n+1}+35691840x_{n+2}\end{array}\right]$
- $\frac{1}{(12312)^3}\left[\begin{array}{l}15035447100672x_{5n+5}-1030783392x_{5n+7}\\+5(99188x_{n+1}-68x_{n+3})^3\\-75177235503360x_{n+1}+51539016960x_{n+3}\end{array}\right]$
- $\frac{1}{(27)^3}\left[\begin{array}{l}80190x_{5n+5}-49572y_{5n+5}\\+5(110x_{n+1}-68y_{n+1})^3-400950x_{n+1}+247860y_{n+1}\end{array}\right]$
- $\frac{1}{(513)^3}\left[\begin{array}{l}1086887970x_{5n+5}-17895492y_{5n+6}+5(4130x_{n+1}-68y_{n+2})^3\\-5434439850x_{n+1}+89477460y_{n+2}\end{array}\right]$
- $\frac{1}{(19467)^3}\left[\begin{array}{l}59432938077870x_{5n+5}-25769558054y_{5n+7}\\+5(156830x_{n+1}-68x_{n+3})^3\\-297164690389 35x_{n+1}+128847790260y_{n+3}\end{array}\right]$

- $\frac{1}{(324)^3}\begin{bmatrix} 10412359488x_{5n+6} - 274197312x_{5n+7} \\ +5(99188x_{n+2} - 2612x_{n+3})^3 \\ -52061797440x_{n+2} + 1370986560x_{n+3} \end{bmatrix}$

- $\frac{1}{(513)^3}\begin{bmatrix} 28948590x_{5n+6} - 687397428y_{5n+5} + 5(110x_{n+2} - 2612y_{n+1})^3 \\ -144742950x_{n+2} + 3436987140y_{n+1} \end{bmatrix}$

- $\frac{1}{(27)^3}\begin{bmatrix} 3010770x_{5n+6} - 1904148y_{5n+6} + 5(4130x_{n+2} - 2612y_{n+2})^3 \\ -15053850x_{n+2} + 9520740y_{n+2} \end{bmatrix}$

- $\frac{1}{(513)^3}\begin{bmatrix} 41272794270x_{5n+6} - 687397428y_{5n+7} \\ +5(156830x_{n+2} - 2612y_{n+3})^3 \\ -206363971350x_{n+2} + 3436987140y_{n+3} \end{bmatrix}$

- $\frac{1}{(19467)^3}\begin{bmatrix} (41686049790x_{5n+7} - 37588690059732y_{5n+5}) \\ +5(110x_{n+3} - 99188y_{n+1})^3 \\ -(208430248950x_{n+3} + 1879434450298660y_{n+1}) \end{bmatrix}$

- $\frac{1}{(513)^3}\begin{bmatrix} (263169x_{5n+7} - 26103206772y_{5n+6}) \\ +5(4130x_{n+3} - 99188y_{n+2})^3 \\ -(5434439850x_{n+3} + 130516033860y_{n+2}) \end{bmatrix}$

- $\frac{1}{(27)^3}\begin{bmatrix} (114329070x_{5n+7} - 72308052y_{5n+7}) \\ +5(156830x_{n+3} - 99188y_{n+3})^3 \\ -571645350x_{n+3} + 3615402060y_{n+3} \end{bmatrix}$

- $\frac{1}{(810)^3}\begin{bmatrix} (7217100y_{5n+6} - 270969300y_{5n+5}) \\ +5(110y_{n+2} - 4130y_{n+1})^3 \\ -36085500y_{n+2} + 3548465000y_{n+1} \end{bmatrix}$

- $\frac{1}{(30780)^3}\begin{bmatrix} 104214924000y_{5n+7} - 14858205937200y_{5n+5} \\ +5(110y_{n+3} - 156830y_{n+3})^3 \\ -521074620000y_{n+1} + 74291029686000y_{n+1} \end{bmatrix}$

- $\frac{1}{(810)^3}\begin{bmatrix} (2709693000y_{5n+7} - 102896163000y_{5n+6}) \\ +5(4130y_{n+3} - 156830y_{n+2})^3 \\ -135484650000y_{n+3} + 514480815000y_{n+2} \end{bmatrix}$

2.5. Each of the following expressions is a Nasty number

- $\frac{1}{324}[15672x_{2n+2} - 408x_{2n+3} + 3888]$
- $\frac{1}{12312}[595128x_{2n+2} - 408x_{2n+4} + 147744]$
- $\frac{1}{27}[660x_{2n+2} - 408y_{2n+2} + 324]$
- $\frac{1}{513}[24780x_{2n+2} - 408y_{2n+3} + 6156]$
- $\frac{1}{19467}[940980x_{2n+2} - 408y_{2n+4} + 233568]$
- $\frac{1}{324}[595128x_{2n+3} - 15672x_{2n+4} + 3888]$
- $\frac{1}{513}[660x_{2n+3} - 15672y_{2n+2} + 6156]$
- $\frac{1}{27}[24780x_{2n+3} - 15672y_{2n+2} + 324]$
- $\frac{1}{513}[940980x_{2n+3} - 15672y_{2n+4} + 6156]$
- $\frac{1}{19467}[660x_{2n+4} - 595128y_{2n+2} + 233604]$
- $\frac{1}{513}[24780x_{2n+4} - 595128y_{2n+3} + 6156]$
- $\frac{1}{27}[940980x_{2n+4} - 595128y_{2n+4} + 324]$
- $\frac{1}{810}[-24780y_{2n+2} + 660y_{2n+3} + 9720]$
- $\frac{1}{30780}[-940980y_{2n+2} + 660y_{2n+4} + 369360]$

- $\frac{1}{810}\left[-940980y_{2n+3}+24780y_{2n+4}+9720\right]$

2.6. Remarkable observation

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table 2: below.

Table 2: Hyperbolas

S. No	Hyperbolas	(X_n, Y_n)
1	$10X_n^2 - Y_n^2 = 4199040$	$\begin{pmatrix}2612x_{n+1}-68x_{n+2}, \\ 220x_{n+2}-8260x_{n+1}\end{pmatrix}$
2	$10X_n^2 - Y_n^2 = 6063413760$	$\begin{pmatrix}99188x_{n+1}-68x_{n+3}, \\ 220x_{n+3}-313660x_{n+1}\end{pmatrix}$
3	$10X_n^2 - Y_n^2 = 23040$	$\begin{pmatrix}110x_{n+1}-68y_{n+1}, \\ 220y_{n+1}-340x_{n+1}\end{pmatrix}$
4	$10X_n^2 - Y_n^2 = 10526760$	$\begin{pmatrix}4130x_{n+1}-68y_{n+2}, \\ 220y_{n+2}-13060x_{n+1}\end{pmatrix}$
5	$10X_n^2 - Y_n^2 = 1515856350$	$\begin{pmatrix}156830x_{n+1}-68y_{n+3}, \\ 220y_{n+3}-495940x_{n+1}\end{pmatrix}$
6	$10X_n^2 - Y_n^2 = 4199040$	$\begin{pmatrix}99188x_{n+2}-2612x_{n+3}, \\ 8260x_{n+3}-313660x_{n+2}\end{pmatrix}$
7	$10X_n^2 - Y_n^2 = 10526760$	$\begin{pmatrix}110x_{n+2}-2612y_{n+1}, \\ 8260y_{n+1}-340x_{n+2}\end{pmatrix}$
8	$10X_n^2 - Y_n^2 = 29160$	$\begin{pmatrix}4130x_{n+2}-2612y_{n+2}, \\ 8260y_{n+2}-13060x_{n+2}\end{pmatrix}$
9	$10X_n^2 - Y_n^2 = 10526760$	$\begin{pmatrix}156830x_{n+2}-2612y_{n+3}, \\ 8260y_{n+3}-495940x_{n+2}\end{pmatrix}$
10	$10X_n^2 - Y_n^2 = 1515856350$	$\begin{pmatrix}110x_{n+3}-99188y_{n+1}, \\ 313660y_{n+1}-340x_{n+3}\end{pmatrix}$
11	$10X_n^2 - Y_n^2 = 10526760$	$\begin{pmatrix}4130x_{n+3}-99188y_{n+1}, \\ 313660y_{n+2}-13060x_{n+3}\end{pmatrix}$
12	$10X_n^2 - Y_n^2 = 29160$	$\begin{pmatrix}156830x_{n+3}-99188y_{n+3}, \\ 313660y_{n+3}-495940x_{n+3}\end{pmatrix}$
13	$10X_n^2 - Y_n^2 = 26244000$	$\begin{pmatrix}110y_{n+2}-4130y_{n+1}, \\ 13060y_{n+1}-340y_{n+2}\end{pmatrix}$
14	$10X_n^2 - Y_n^2 = 3789633600$	$\begin{pmatrix}110y_{n+3}-156830y_{n+1}, \\ 495940y_{n+1}-340y_{n+3}\end{pmatrix}$
15	$10X_n^2 - Y_n^2 = 2624000$	$\begin{pmatrix}810y_{n+3}-156830y_{n+2}, \\ 495940y_{n+2}-13060y_{n+3}\end{pmatrix}$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table 3: below

Table 3: Parabolas

S. No	Parabolas	(X_n, Y_n)
1	$3240X_n - Y_n^2 = 4199040$	$\begin{pmatrix}2612x_{2n+2}-68x_{2n+3}+648, \\ 220x_{n+2}-8260x_{n+1}\end{pmatrix}$

2	$123120X_n - Y_n^2 = 606341376$	$\begin{pmatrix} 99188x_{2n+2} - 68x_{2n+4} + 24624, \\ 220x_{n+3} - 313660x_{n+1} \end{pmatrix}$
3	$270X_n - Y_n^2 = 230402916$	$\begin{pmatrix} 110x_{2n+2} - 68y_{2n+2} + 54, \\ 220y_{n+1} - 340x_{n+1} \end{pmatrix}$
4	$5130X_n - Y_n^2 = 10526760$	$\begin{pmatrix} 4130x_{2n+2} - 68y_{2n+3} + 1026, \\ 220y_{n+2} - 13060x_{n+1} \end{pmatrix}$
5	$194670X_n - Y_n^2 = 15158563560$	$\begin{pmatrix} 156830x_{2n+2} - 68y_{2n+4} + 38928, \\ 220y_{n+3} - 495940x_{n+1} \end{pmatrix}$
6	$3240X_n - Y_n^2 = 419904$	$\begin{pmatrix} 99188x_{2n+3} - 2612x_{2n+4} + 648, \\ 8260x_{n+3} - 313660x_{n+2} \end{pmatrix}$
7	$5130X_n - Y_n^2 = 10526760$	$\begin{pmatrix} 110x_{2n+3} - 2612y_{2n+2} + 1026, \\ 8260y_{n+1} - 340x_{n+2} \end{pmatrix}$
8	$270X_n - Y_n^2 = 29160$	$\begin{pmatrix} 4130x_{2n+3} - 2612y_{2n+3} + 54, \\ 8260y_{n+2} - 13060x_{n+2} \end{pmatrix}$
9	$5130X_n - Y_n^2 = 10526760$	$\begin{pmatrix} 156830x_{2n+3} - 2612y_{2n+4} + 1026, \\ 8260y_{n+3} - 495940x_{n+2} \end{pmatrix}$
10	$194670X_n - Y_n^2 = 15158563560$	$\begin{pmatrix} 110x_{2n+4} - 99188y_{2n+2} + 3834, \\ 313660y_{n+1} - 340x_{n+3} \end{pmatrix}$
11	$5130X_n - Y_n^2 = 10526760$	$\begin{pmatrix} 4130x_{2n+4} - 99188y_{2n+2} + 1026, \\ 313660y_{n+2} - 13060x_{n+3} \end{pmatrix}$
12	$270X_n - Y_n^2 = 29160$	$\begin{pmatrix} 156830x_{2n+4} - 99188y_{2n+4} + 54, \\ 313660y_{n+3} - 495940x_{n+3} \end{pmatrix}$
13	$8100X_n - Y_n^2 = 26244000$	$\begin{pmatrix} 110y_{2n+3} - 4130y_{2n+2} + 1620, \\ 13060y_{n+1} - 340y_{n+2} \end{pmatrix}$
14	$307800X_n - Y_n^2 = 3789633600$	$\begin{pmatrix} 110y_{2n+4} - 156830y_{2n+2} + 61560, \\ 495940y_{n+1} - 340y_{n+3} \end{pmatrix}$
15	$8100X_n - Y_n^2 = 2624000$	$\begin{pmatrix} 810y_{2n+4} - 156830y_{2n+3} + 1620, \\ 495940y_{n+2} - 13060y_{n+3} \end{pmatrix}$

2.7.

Consider $p = x + y, q = y$ .Observe that $p > q > 0$. Treat p, q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$,

$$\alpha = 2pq\,,\ \beta = p^2 - q^2,\ \gamma = p^2 + q^2$$

Then the following interesting relations are observed:

a) $Y - 5X + 4Z = 27$

b) $\frac{2A}{P} = x_{n+1}.y_{n+1}$

c) $3\left(X - \frac{4A}{P}\right)$ is a Nasty number

d) $Z - X$ is a Perfect square

2.8. Note :

Instead of (2) , we replace x by $x = X - 2T$ and y by $y = X - 5T$ (11)

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution (T_{n+1}, X_{n+1}) of (3) is found to be

Using (7) and (8) in (11), we have

$$x_{n+1} = X_{n+1} - 2T_{n+1} = \frac{3}{2} f_n + \frac{6}{2\sqrt{10}} g_n \qquad (12)$$

$$y_{n+1} = X_{n+1} - 5T_{n+1} = -\frac{3}{2} f_n - \frac{15}{2\sqrt{10}} g_n \qquad (13)$$

Thus, (12) and (13) represent the integer solutions of the hyperbola (1)

A few numerical examples are given in the following Table 4

Table 4: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	3	-3
0	93	-147
1	3531	-5583
2	134085	-212007

3. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the Diophantine equations represented by hyperbola is given by $5x^2 - 2y^2 = 27$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their solutions among the suitable properties.

REFERENCES

[1] MollinRA, Anithasrinivasan . A note on the negative pellEquation,International Journal of Algebra,2010;4(19):919-922

[2] WhitfordEE.Some solutions of the Pellian Equation $x^2 - Ay^2 = \pm 4$ JSTOR:Annals of Mathematics, Second series, 1913-1914;1:157-160.

[3] AhmetTekcan S, Bet W Gezer, Osman Bizim.On the Integer solutions of the pellEquations $x^2 - dy^2 = 2^t$, World Academy of Science, Engineering and Technology, 2007;1:522-526.

[4] AhmetTekcan. The Pell Equation $x^2 - (k^2 - k)y^2 = 2^t$.World Academy of Science, Engineering and Technology, 2008;19:697-701.

[5] MerveGuney. The Pell Equation $x^2 - (a^2b^2 + 2b)y^2 = 2^t$, When $N \in (\pm 1, \pm 4)$, Mathematica Aterna.2012;2(7):629-701.

[6] Sangeetha V, GopalanMA,ManjuSomanath. On the Integral Solutions of the Pell Equation $x^2 = 13y^2 - 3^t$, International Journal of Applied Mathematical Research.2014;3(1):58-61.

[7] Gopalan MA,Sumathi G, Vidhyalakshmi S. Observations on the hyperbola $x^2 = 19y^2 - 3^t$,Journal of the Engineering and Technology, 2014;2(2A):152-155.

[8] Gopalan MA, Vidhyalakshmi S, Kavitha A. On the Integral solution of the binary quadratic equation $x^2 = 15y^2 - 11^2$,Scholars Journal of the Engineering and Technology,2014;2(2A)156-158.

[9] VidhyalakshmiS,KrithikaV,AgalyaK.On the negative Pell Equation, Proceeding of the National Conference on MATAM, 2015,4-9.

[10] MeenaK,GopalanMA,KarthikaR,On the Negative Pell Equation $y^2 = 10x^2 - 6$,Integral Journal of Multidisciptinary Research and Development.2015;2(12):390-392.

[11] Gopalan MA, Vidhyalakshmi S, Pandichelvi V, Sivakamasundari P, Priyadharsini C. On the Negative pellequation $y^2 = 45x^2 - 11$,International Journal of pure Mathematical Science.2016;16:30-36.

[12] Meena K, VidhyalakshmiS,Rukmani A. On the Negative pellequation $y^2 = 31x^2 - 6$,Universe of Engineering Technologies and Science.2015;11(X11):1-4.

[13] Meena K,Gopalan MA, Bhuvaneswari E.On the Negative pellequation $y^2 = 60x^2 - 15$,Scholars Bulletin,2015;1(11):310-316.

[14] Gopalan MA, Vidhyalakshmi S, PresennaR,Vanitha M. Observation On the Negative pell equation $y^2 = 180x^2 - 11$.Universal Journal of Mathematics.2(1):41-45.

[15] Vidhyalakshmi S,Gopalan MA, Premalatha E, Sofia ChristinalS.On the Negative Pell Equation $y^2 = 72x^2 - 8$.International Journal of Engineering Research (IJETER).2016;4(2):25-28.

[16] Gopalan MA, Vidhyalakshmi S, ThiruniraiselviN.Astudy on the Hyperbola $y^2 = 8x^2 - 31$.International Journal of latest Research in Science and Technology,2013;2(1):454-456.